

GENERALIZED (κ, μ) -SPACE FORMS

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ABSTRACT. Generalized (κ, μ) -space forms are introduced and studied. We examine in depth the contact metric case and present examples for all possible dimensions. We also analyse the trans-Sasakian case.

1. INTRODUCTION

A generalized Sasakian space form was defined by the first named author (jointly with P. Alegre and D. E. Blair) in [1] as that almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor R is given by

$$(1.1) \quad R = f_1 R_1 + f_2 R_2 + f_3 R_3,$$

where f_1, f_2, f_3 are some differentiable functions on M and

$$\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi, \end{aligned}$$

for any vector fields X, Y, Z on M . We denote it by $M(f_1, f_2, f_3)$.

Since then, several papers have appeared concerning different aspects of this topic: structures, submanifolds and conformal changes of metric ([2], [3] and [4]), B.-Y. Chen's inequalities ([5]), slant submanifolds inheriting the structure ([6]), CR-submanifolds ([7] and [8]), the Ricci curvature of some submanifolds ([19], [24] and [27]), conformal flatness and local symmetry ([20]), some other symmetry properties ([17] and [18]) and immersions of warped products ([25] and [28]). Other related papers are [15] and [16].

On the other hand, a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be a generalized (κ, μ) -space if its curvature tensor satisfies the condition

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\},$$

for some smooth functions κ and μ on M independent of the choice of vectors fields X and Y . If κ and μ are constant, the manifold is called a (κ, μ) -space. T. Koufogiorgos proved in [21] that if a (κ, μ) -space M has constant ϕ -sectional curvature c and dimension greater than 3, the curvature tensor of this (κ, μ) -space

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form is given by

$$(1.2) \quad R = \frac{c+3}{4} R_1 + \frac{c-1}{4} R_2 + \left(\frac{c+3}{4} - \kappa \right) R_3 + R_4 + \frac{1}{2} R_5 + (1-\mu) R_6,$$

where R_1, R_2, R_3 are the tensors defined above and

$$\begin{aligned} R_4(X, Y)Z &= g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y, \\ R_5(X, Y)Z &= g(hY, Z)hX - g(hX, Z)hY \\ &\quad + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX, \\ R_6(X, Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \\ &\quad + g(hX, Z)\eta(Y)\xi - g(hY, Z)\eta(X)\xi, \end{aligned}$$

for any vector fields X, Y, Z , where $2h = L_\xi \phi$ and L is the usual Lie derivative. Now, a natural question arises: is it possible to generalize the notion of (κ, μ) -space form, by replacing the constants in equation (1.2) by some differentiable functions?

Therefore, in view of the previous works on generalized Sasakian space forms, we say that an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is a *generalized (κ, μ) -space form* if there exist differentiable functions $f_1, f_2, f_3, f_4, f_5, f_6$ on M such that

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6.$$

It is obvious that (κ, μ) -space forms are natural examples of generalized (κ, μ) -space forms, with constant functions

$$f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - \kappa, f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1 - \mu.$$

We also have generalized Sasakian space forms, with $f_4 = f_5 = f_6 = 0$.

Thus, in this paper we introduce and study generalized (κ, μ) -space forms. The paper is organized as follows. The section 2 contains some necessary background on almost contact metric geometry. Afterwards, in section 3 we formally give the definition of generalized (κ, μ) -space form and check that some results that were true for generalized Sasakian space forms are also correct for generalized (κ, μ) -space forms. Then, we obtain some basic identities for generalized (κ, μ) -space forms which are analogous to those satisfied by Sasakian manifolds. In section 4 we prove that contact metric generalized (κ, μ) -space forms are generalized (κ, μ) -spaces with $\kappa = f_1 - f_3$ and $\mu = f_4 - f_6$. Next, we observe that if dimension is greater than or equal to 5, then they are $(-f_6, 1 - f_6)$ -spaces with constant ϕ -sectional curvature $2f_6 - 1$. Furthermore, $f_4 = 1$, $f_5 = 1/2$ and f_1, f_2, f_3 depend linearly on the constant f_6 . We also give a method for constructing infinitely many examples of this type. Later, we pay attention to the 3-dimensional case, in which we prove that the expression for the curvature tensor is not unique and that several properties and results must be satisfied. We also check that the example of generalized (κ, μ) -space with non-constant κ and μ that T. Koufogiorgos and C. Tsihlias provided in [22] is a generalized (κ, μ) -space form with non-constant functions f_1, f_3 and f_4 . Finally, we prove in section 5 that if a manifold is trans-Sasakian, then $h = 0$. Therefore, generalized (κ, μ) -space forms with trans-Sasakian structure are generalized Sasakian space forms, already studied in [2].

2. PRELIMINARIES

In this section, we recall some general definitions and basic formulas which will be used later. For more background on almost contact metric manifolds, we recommend the reference [9]. Anyway, we will recall some more specific notions and results in the following sections, when required.

An odd-dimensional Riemannian manifold (M, g) is said to be an *almost contact metric manifold* if there exist on M a $(1, 1)$ -tensor field ϕ , a vector field ξ (called the *structure vector field*) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2 X = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X, Y on M . In particular, on an almost contact metric manifold we also have $\phi\xi = 0$ and $\eta \circ \phi = 0$.

Such a manifold is said to be a *contact metric manifold* if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is the *fundamental 2-form* of M . If, in addition, ξ is a Killing vector field, then M is said to be a *K-contact manifold*. It is well-known that a contact metric manifold is a *K-contact manifold* if and only if

$$(2.1) \quad \nabla_X \xi = -\phi X$$

for all vector fields X on M . Even an almost contact metric manifold satisfying the equation (2.1) becomes a *K-contact manifold*.

On the other hand, the almost contact metric structure of M is said to be *normal* if the Nijenhuis torsion $[\phi, \phi]$ of ϕ equals $-2d\eta \otimes \xi$. A normal contact metric manifold is called a *Sasakian manifold*. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(2.2) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

for any vector fields X, Y on M . Moreover, for a Sasakian manifold the following equation holds:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

Given an almost contact metric manifold (M, ϕ, ξ, η, g) , a ϕ -section of M at $p \in M$ is a section $\Pi \subseteq T_p M$ spanned by a unit vector X_p orthogonal to ξ_p , and ϕX_p . The ϕ -sectional curvature of Π is defined by $K(X, \phi X) = R(X, \phi X, \phi X, X)$. A Sasakian manifold with constant ϕ -sectional curvature c is called a *Sasakian space form*. In such a case, its Riemann curvature tensor is given by equation (1.1) with functions $f_1 = (c + 3)/4$ and $f_2 = f_3 = (c - 1)/4$.

It is well known that on a contact metric manifold (M, ϕ, ξ, η, g) , the tensor h , defined by $2h = L_\xi \phi$, is symmetric and satisfies the following relations [9]

$$(2.3) \quad h\xi = 0, \quad \nabla_X \xi = -\phi X - \phi hX, \quad h\phi = -\phi h, \quad \text{tr}(h) = 0, \quad \eta \circ h = 0.$$

Therefore, it follows from equations (2.1) and (2.3) that a contact metric manifold is *K-contact* if and only if $h = 0$.

Finally, we assume that all the functions considered in this paper will be differentiable functions on the corresponding manifolds.

3. DEFINITION AND FIRST RESULTS

In this section we give the formal definition of generalized (κ, μ) -space forms and prove some basic results about these manifolds. Then we study some interesting properties of their curvature tensor.

Definition 3.1. We say that an almost contact metric manifold (M, ϕ, ξ, η, g) is a *generalized (κ, μ) -space form* if there exist functions $f_1, f_2, f_3, f_4, f_5, f_6$ defined on M such that

$$(3.1) \quad R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6,$$

where $R_1, R_2, R_3, R_4, R_5, R_6$ are the following tensors

$$\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi, \\ R_4(X, Y)Z &= g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y, \\ R_5(X, Y)Z &= g(hY, Z)hX - g(hX, Z)hY \\ &\quad + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX, \\ R_6(X, Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \\ &\quad + g(hX, Z)\eta(Y)\xi - g(hY, Z)\eta(X)\xi, \end{aligned}$$

for all vector fields X, Y, Z on M , where $2h = L_\xi \phi$ and L is the usual Lie derivative. We will denote such a manifold by $M(f_1, \dots, f_6)$.

Remark 3.2. It is obvious that (κ, μ) -contact space forms of dimension greater than 3 are natural examples of generalized (κ, μ) -space forms, where

$$f_1 = \frac{c+3}{4}, \quad f_2 = \frac{c-1}{4}, \quad f_3 = \frac{c+3}{4} - \kappa, \quad f_4 = 1, \quad f_5 = \frac{1}{2}, \quad f_6 = 1 - \mu$$

are constant. Generalized Sasakian space forms (defined in [1]) are also examples with $f_4 = f_5 = f_6 = 0$ and f_1, f_2, f_3 not necessarily constant.

As we have already pointed out, $h = 0$ for a K -contact manifold. Therefore, a generalized (κ, μ) -space form with such a structure is actually a generalized Sasakian space form. Hence, the following results are inferred from Proposition 3.6, Theorem 3.7 and Theorem 3.15 from [1]:

Theorem 3.3. *Let $M(f_1, \dots, f_6)$ be a generalized (κ, μ) -space form. If M is a K -contact manifold, then $f_3 = f_1 - 1$. Moreover, M is Sasakian.*

Theorem 3.4. *Let $M(f_1, \dots, f_6)$ be a generalized (κ, μ) -space form. If M is a Sasakian manifold, then $f_2 = f_3 = f_1 - 1$.*

Remark 3.5. Sasakian manifolds are always K -contact, while the converse is not true in general, only in dimension 3. However, we have just seen that being Sasakian and K -contact are equivalent concepts for generalized (κ, μ) -space forms.

Using the properties of h , it can be proved that:

Theorem 3.6. *Let $M(f_1, \dots, f_6)$ be a generalized (κ, μ) -space form. If M is a contact metric manifold with $f_3 = f_1 - 1$, then it is a Sasakian manifold.*

Proof. Let M^{2n+1} be a contact metric manifold satisfying $f_3 = f_1 - 1$. Because of Theorem 3.3 we would only need to prove that M is K -contact, which is equivalent to checking that $S(\xi, \xi) = 2n$, where S denotes the Ricci curvature tensor (see [9], p. 92).

If we take a local orthonormal basis $\{e_1, \dots, e_{2n}, \xi\}$, then a direct computation from (3.1) gives

$$R(e_i, \xi, \xi, e_i) = 1 + (f_4 - f_6)g(he_i, e_i),$$

where we have used the properties of almost contact metric manifolds, the fact that $h\xi = 0$ and the hypothesis $f_1 - f_3 = 1$. Therefore,

$$\begin{aligned} S(\xi, \xi) &= \sum_{i=1}^{2n} R(e_i, \xi, \xi, e_i) + R(\xi, \xi, \xi, \xi) = \sum_{i=1}^{2n} (1 + (f_4 - f_6)g(he_i, e_i)) \\ &= 2n + (f_4 - f_6) \sum_{i=1}^{2n} g(he_i, e_i) = 2n + (f_4 - f_6)\text{tr}(h) = 2n, \end{aligned}$$

because of (2.3). \square

We will now calculate $K(X, \phi X)$ (the ϕ -sectional curvature), $K(X, \xi)$ (the ξ -sectional curvature) and $K(\phi X, \xi)$ for generalized (κ, μ) -space forms.

Proposition 3.7. *Let $M(f_1, \dots, f_6)$ be a generalized (κ, μ) -space form. Then the ϕ -sectional curvature of the ϕ -section spanned by the unit vector field X , orthogonal to ξ , is given by*

$$K(X, \phi X) = f_1 + 3f_2 + f_4 g((h - \phi h \phi)X, X).$$

If M is also a contact metric manifold, then the ϕ -sectional curvature is given by

$$K(X, \phi X) = f_1 + 3f_2,$$

so it does not depend on the choice of vector field X .

Proof. By applying the properties of an almost contact metric manifold, we can directly calculate from (3.1) the ϕ -sectional curvature of the ϕ -section spanned by $\{X, \phi X\}$, where X is a unit vector field orthogonal to ξ , as follows:

$$\begin{aligned} K(X, \phi X) &= f_1 + 3f_2 + f_4 \{g(hX, X) + g(h\phi X, \phi X)\} \\ &= f_1 + 3f_2 + f_4 g((h - \phi h \phi)X, X). \end{aligned}$$

If M is a contact metric manifold then, in view of (2.3), $h\phi = -\phi h$ and we get

$$(h - \phi h \phi)X = (h + h\phi^2)X = h(X + \phi^2 X) = h(X - X) = 0.$$

Therefore, in such a case, $K(X, \phi X) = f_1 + 3f_2$. \square

A direct computation, similar to that of Theorem 3.6, gives:

Proposition 3.8. *Let $M(f_1, \dots, f_6)$ be a generalized (κ, μ) -space form. Then the ξ -sectional curvature of the ξ -section spanned by the unit vector field X , orthogonal to ξ , is given by*

$$K(X, \xi) = f_1 - f_3 + (f_4 - f_6)g(hX, X).$$

Corollary 3.9. *Let $M(f_1, \dots, f_6)$ be a generalized (κ, μ) -space form. If X is a unit vector field orthogonal to ξ , then*

$$K(\phi X, \xi) = f_1 - f_3 + (f_4 - f_6)g(h\phi X, \phi X).$$

If M is also a contact metric manifold, then

$$K(\phi X, \xi) = f_1 - f_3 - (f_4 - f_6)g(hX, X).$$

Proof. The first assertion follows directly from Proposition 3.8. With respect to the contact metric case, we just have to take into account that

$$g(h\phi X, \phi X) = -g(\phi hX, \phi X) = -g(hX, X),$$

by virtue of (2.3). \square

The following classic result appears in ([9], pp. 94-95):

Lemma 3.10. *Let M be a Sasakian manifold. If we put*

$$\begin{aligned} \tilde{P}(X, Y, Z, W) &= d\eta(X, Z)g(Y, W) - d\eta(X, W)g(Y, Z) \\ &\quad - d\eta(Y, Z)g(X, W) + d\eta(Y, W)g(X, Z), \end{aligned}$$

then

$$(3.2) \quad R(X, Y, Z, \phi W) + R(X, Y, \phi Z, W) = -\tilde{P}(X, Y, Z, W)$$

for any vector fields X, Y, Z, W on M and

$$(3.3) \quad R(\phi X, \phi Y, \phi Z, \phi W) = R(X, Y, Z, W),$$

$$(3.4) \quad R(X, \phi X, Y, \phi Y) = R(X, Y, X, Y) + R(X, \phi Y, X, \phi Y) - 2\tilde{P}(X, Y, X, \phi Y)$$

for any vector fields X, Y, Z, W orthogonal to ξ .

Let (M, ϕ, ξ, η, g) be any almost contact metric manifold. We now denote

$$\begin{aligned} P(X, Y, Z, W) &= g(X, \phi Z)g(Y, W) - g(X, \phi W)g(Y, Z) \\ &\quad - g(Y, \phi Z)g(X, W) + g(Y, \phi W)g(X, Z) \end{aligned}$$

for any vectors fields X, Y, Z, W on M . In particular, if M is a contact metric manifold, $P = \tilde{P}$. We will study whether similar results hold true for generalized (κ, μ) -space forms.

We omit the corresponding proofs because they can be easily obtained by making direct computations from (3.1) and using (2.3) in the contact metric cases.

Proposition 3.11. *Let $M(f_1, \dots, f_6)$ be a generalized (κ, μ) -space form. Given X, Y, Z, W orthogonal to ξ , we have*

$$R_i(\phi X, \phi Y, \phi Z, \phi W) = R_i(X, Y, Z, W) \quad \text{for } i = 1, 2, 3, 6.$$

Therefore,

$$\begin{aligned} R(\phi X, \phi Y, \phi Z, \phi W) &- f_4 R_4(\phi X, \phi Y, \phi Z, \phi W) - f_5 R_5(\phi X, \phi Y, \phi Z, \phi W) \\ &= R(X, Y, Z, W) - f_4 R_4(X, Y, Z, W) - f_5 R_5(X, Y, Z, W). \end{aligned}$$

If M is also a contact metric manifold, then

$$\begin{aligned} R_4(\phi X, \phi Y, \phi Z, \phi W) &= -R_4(X, Y, Z, W), \\ R_5(\phi X, \phi Y, \phi Z, \phi W) &= R_5(X, Y, Z, W), \end{aligned}$$

and therefore

$$R(\phi X, \phi Y, \phi Z, \phi W) = R(X, Y, Z, W) - 2f_4 R_4(X, Y, Z, W).$$

Proposition 3.12. *Let $M(f_1, \dots, f_6)$ be a generalized (κ, μ) -space form. Then*

$$\begin{aligned} R(X, Y, Z, \phi W) + R(X, Y, \phi Z, W) &= -(f_1 - f_2)P(X, Y, Z, W) \\ &\quad - f_4(P(hX, Y, Z, W) + P(X, hY, Z, W)) - 2f_5 P(hX, hY, Z, W) \end{aligned}$$

for any vector fields X, Y, Z, W orthogonal to ξ .

It is clear that the above results extend Propositions 3.14 and 3.17 from [1]. To obtain a result similar to equation (3.4), we need the manifold to be a contact metric one:

Proposition 3.13. *Let $M(f_1, \dots, f_6)$ be a generalized (κ, μ) -space form. If M is also a contact metric manifold, then*

$$\begin{aligned} R(X, \phi X, Y, \phi Y) = & R(X, Y, X, Y) + R(X, \phi Y, X, \phi Y) \\ & - 2(f_1 - f_2)P(X, Y, X, \phi Y) - 2f_4P(X, Y, hX, \phi Y), \end{aligned}$$

for any vector fields X, Y orthogonal to ξ .

4. CONTACT METRIC GENERALIZED (κ, μ) -SPACE FORMS

In this section we will study contact metric generalized (κ, μ) -space forms. The first fundamental fact is that such a manifold is a generalized (κ, μ) -space.

Theorem 4.1. *If $M(f_1, \dots, f_6)$ is a contact metric generalized (κ, μ) -space form, then it is a generalized (κ, μ) -space, with $\kappa = f_1 - f_3$ and $\mu = f_4 - f_6$.*

Proof. Using the definition of the tensors R_1, R_2 and R_3 , we obtain by direct computation that for every X, Y vector fields on M :

$$\begin{aligned} R_1(X, Y)\xi &= -R_3(X, Y)\xi = \eta(Y)X - \eta(X)Y, \\ R_2(X, Y)\xi &= 0. \end{aligned}$$

Moreover, by the properties (2.3) of the tensor h and the definition of R_4, R_5, R_6 , it also holds that:

$$\begin{aligned} R_4(X, Y)\xi &= -R_6(X, Y)\xi = \eta(Y)hX - \eta(X)hY, \\ R_5(X, Y)\xi &= 0. \end{aligned}$$

Therefore, it would be enough to use the formula (3.1) to obtain that the curvature tensor of a generalized (κ, μ) -space form satisfies

$$R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\},$$

for every X, Y . □

We know from Theorem 3.6 that $M(f_1, \dots, f_6)$ is Sasakian if $\kappa = f_1 - f_3 = 1$. Under the same hypotheses, we also know that $f_2 = f_3$ (Theorem 3.4) and $h = 0$, so we may take $f_4 = f_5 = f_6 = 0$. Therefore, in the remainder of this section we will study non-Sasakian generalized (κ, μ) -space forms $M(f_1, \dots, f_6)$, that is, those with $\kappa = f_1 - f_3 \neq 1$.

We will use the following result from [10]:

Theorem 4.2. *If M is a (κ, μ) -space, then $\kappa \leq 1$. If $\kappa = 1$, then $h = 0$ and M is a Sasakian manifold. If $\kappa < 1$, then M admits three mutually orthogonal and integrable distributions $\mathcal{D}(0), \mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ determined by the eigenspaces of h ,*

where $\lambda = \sqrt{1 - \kappa}$. Moreover,

$$(4.1) \quad R(X_\lambda, Y_\lambda)Z_{-\lambda} = (\kappa - \mu)\{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\},$$

$$(4.2) \quad R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = (\kappa - \mu)\{g(\phi Y_{-\lambda}, Z_\lambda)\phi X_{-\lambda} - g(\phi X_{-\lambda}, Z_\lambda)\phi Y_{-\lambda}\},$$

$$(4.3) \quad R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = \kappa g(\phi X_\lambda, Z_{-\lambda})\phi Y_{-\lambda} + \mu g(\phi X_\lambda, Y_{-\lambda})\phi Z_{-\lambda},$$

$$(4.4) \quad R(X_\lambda, Y_{-\lambda})Z_\lambda = -\kappa g(\phi Y_{-\lambda}, Z_\lambda)\phi X_\lambda - \mu g(\phi Y_{-\lambda}, X_\lambda)\phi Z_\lambda,$$

$$(4.5) \quad R(X_\lambda, Y_\lambda)Z_\lambda = (2(1 + \lambda) - \mu)\{g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda\},$$

$$(4.6) \quad R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (2(1 - \lambda) - \mu)\{g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}\}.$$

The following results appear in [21]:

Theorem 4.3. *Let M be a (κ, μ) -space of dimension greater than or equal to 5. If the ϕ -sectional curvature at any point of M is independent of the choice of the ϕ -section at that point, then it is constant on M and the curvature tensor is given by*

$$(4.7) \quad R = \frac{c+3}{4}R_1 + \frac{c-1}{4}R_2 + \left(\frac{c+3}{4} - \kappa\right)R_3 + R_4 + \frac{1}{2}R_5 + (1-\mu)R_6,$$

where c is the constant ϕ -sectional curvature. Moreover, if $\kappa \neq 1$, then $\mu = \kappa + 1$ and $c = -2\kappa - 1$.

Theorem 4.4. *Let M be a non-Sasakian (κ, μ) -space of dimension greater than or equal to 5. Then M has constant ϕ -sectional curvature if and only if $\mu = \kappa + 1$.*

Furthermore, in [22] the following two theorems have been proved:

Theorem 4.5. *Let M be a non-Sasakian generalized (κ, μ) -space of dimension greater than or equal to 5. Then, the functions κ, μ are constant, that is, M is a (κ, μ) -space.*

Theorem 4.6. *Let M be a non-Sasakian generalized (κ, μ) -space. If κ, μ satisfy the condition $a\kappa + b\mu = c$ (where a, b, c are constant), then κ, μ are constant.*

Applying the previous theorems to contact metric generalized (κ, μ) -space forms $M(f_1, \dots, f_6)$, we deduce:

Theorem 4.7. *If $M(f_1, \dots, f_6)$ is a non-Sasakian, contact metric generalized (κ, μ) -space form and $a(f_1 - f_3) + b(f_4 - f_6) = c$, (a, b, c constant), then $f_1 - f_3$ and $f_4 - f_6$ are constant.*

We can also obtain the following theorem, in which we prove that the functions of a contact metric generalized (κ, μ) -space form $M(f_1, \dots, f_6)$ of dimension greater than or equal to 5 are constant and are related. We also obtain a kind of converse result.

Theorem 4.8. *If $M(f_1, \dots, f_6)$ is a non-Sasakian, contact metric generalized (κ, μ) -space form of dimension greater than or equal to 5, then M has constant ϕ -sectional curvature $c = 2f_6 - 1 > -3$ and*

$$(4.8) \quad \begin{aligned} f_1 &= \frac{f_6 + 1}{2}, & f_2 &= \frac{f_6 - 1}{2}, & f_3 &= \frac{3f_6 + 1}{2}, \\ f_4 &= 1, & f_5 &= \frac{1}{2}, & f_6 &= \text{constant} > -1, \\ \kappa &= -f_6 < 1, & \mu &= 1 - f_6 < 2, \end{aligned}$$

Hence M is a $(-f_6, 1 - f_6)$ -space with constant ϕ -sectional curvature $c = 2f_6 - 1 > -3$.

Conversely, let M be a $(-f_6, 1 - f_6)$ -space of dimension greater than or equal to 5 and constant ϕ -sectional curvature $c = 2f_6 - 1 > -3$. Then M is a non-Sasakian, contact metric generalized (κ, μ) -space form $M(f_1, \dots, f_6)$ with constant functions f_1, \dots, f_6 satisfying (4.8).

Proof. In view of Theorems 4.1 and 4.5, it follows that M is a generalized $(f_1 - f_3, f_4 - f_6)$ -space and that $f_1 - f_3, f_4 - f_6$ are constant. Therefore, M is a $(f_1 - f_3, f_4 - f_6)$ -space and applying Theorem 4.2 we obtain that the curvature tensor satisfies equations (4.1)–(4.6) for $\kappa = f_1 - f_3, \mu = f_4 - f_6, X_\lambda, Y_\lambda, Z_\lambda \in \mathcal{D}(\lambda)$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in \mathcal{D}(-\lambda)$. In particular

$$\begin{aligned} R(X_\lambda, Y_\lambda)Z_{-\lambda} &= (f_1 - f_3 - f_4 + f_6) \times \\ &\quad \times \{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\}, \\ R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= (f_1 - f_3)g(\phi X_\lambda, Z_{-\lambda})\phi Y_{-\lambda} \\ &\quad + (f_4 - f_6)g(\phi X_\lambda, Y_{-\lambda})\phi Z_{-\lambda}, \\ R(X_\lambda, Y_\lambda)Z_\lambda &= (2(1 + \sqrt{1 - f_1 + f_3}) - f_4 + f_6) \times \\ &\quad \times \{g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda\}, \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (2(1 - \sqrt{1 - f_1 + f_3}) - f_4 + f_6) \times \\ &\quad \times \{g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}\}. \end{aligned}$$

On the other hand, using the definition of generalized (κ, μ) -space form and the properties of contact metric manifolds we get:

$$\begin{aligned} R(X_\lambda, Y_\lambda)Z_{-\lambda} &= (f_2 - f_5(1 - f_1 + f_3)) \times \\ &\quad \times \{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\}, \\ R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(f_2 + f_5(1 - f_1 + f_3))g(\phi X_\lambda, Z_{-\lambda})\phi Y_{-\lambda} \\ &\quad - 2f_2g(\phi X_\lambda, Y_{-\lambda})\phi Z_{-\lambda} \\ &\quad + (f_1 - f_5(1 - f_1 + f_3))g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\ R(X_\lambda, Y_\lambda)Z_\lambda &= \left(f_1 + f_5(1 - f_1 + f_3) + 2f_4\sqrt{1 - f_1 + f_3}\right) \times \\ &\quad \times \{g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda\}, \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= \left(f_1 + f_5(1 - f_1 + f_3) - 2f_4\sqrt{1 - f_1 + f_3}\right) \times \\ &\quad \times \{g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}\}. \end{aligned}$$

Combining both sets of equations we can write:

$$\begin{aligned} &(f_1 - f_3 - f_4 + f_6)\{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\} \\ (4.9) \quad &= (f_2 - f_5(1 - f_1 + f_3))\{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\}, \end{aligned}$$

$$\begin{aligned} &(f_1 - f_3)g(\phi X_\lambda, Z_{-\lambda})\phi Y_{-\lambda} + (f_4 - f_6)g(\phi X_\lambda, Y_{-\lambda})\phi Z_{-\lambda} \\ &= -(f_2 + f_5(1 - f_1 + f_3))g(\phi X_\lambda, Z_{-\lambda})\phi Y_{-\lambda} - 2f_2g(\phi X_\lambda, Y_{-\lambda})\phi Z_{-\lambda} \\ (4.10) \quad &+ (f_1 - f_5(1 - f_1 + f_3))g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \end{aligned}$$

$$\begin{aligned}
& (2(1+\sqrt{1-f_1+f_3})-f_4+f_6)\{g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda\} \\
& = \left(f_1 + f_5(1-f_1+f_3) + 2f_4\sqrt{1-f_1+f_3}\right) \times \\
(4.11) \quad & \times \{g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda\},
\end{aligned}$$

$$\begin{aligned}
& (2(1-\sqrt{1-f_1+f_3})-f_4+f_6)\{g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}\} \\
& = \left(f_1 + f_5(1-f_1+f_3) - 2f_4\sqrt{1-f_1+f_3}\right) \times \\
(4.12) \quad & \times \{g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}\}.
\end{aligned}$$

The dimension is greater than or equal to 5, so we can take two mutually orthogonal unit vector fields $X_\lambda, Y_\lambda \in \mathcal{D}(\lambda)$ in the equation (4.9) and choosing $Z_{-\lambda} = \phi X_\lambda$ we obtain

$$(f_1 - f_3 - f_4 + f_6)(-\phi Y_\lambda) = (f_2 - f_5(1 - f_1 + f_3))(-\phi Y_\lambda),$$

and we deduce that

$$(4.13) \quad f_1 - f_3 - f_4 + f_6 = f_2 - f_5(1 - f_1 + f_3).$$

From (4.10) we obtain three equations, depending on the choice we make. If we take two mutually orthogonal unit vector fields $Y_{-\lambda}, Z_{-\lambda} \in \mathcal{D}(-\lambda)$ and pick $X_\lambda = \phi Z_{-\lambda}$, we obtain:

$$(4.14) \quad -(f_1 - f_3) = f_2 + f_5(1 - f_1 + f_3).$$

For $Y_{-\lambda}, Z_{-\lambda} \in \mathcal{D}(-\lambda)$ orthogonal and unit and $X_\lambda = \phi Y_{-\lambda}$, it follows that

$$(4.15) \quad -(f_4 - f_6) = 2f_2.$$

If we now take $Y_{-\lambda} = Z_{-\lambda}$ unit and $X_\lambda = \phi Y_{-\lambda}$, we have the following equation:

$$(4.16) \quad -(f_1 - f_3) - (f_4 - f_6) = f_1 + 3f_2.$$

In (4.11) we choose two unit vector fields $Y_\lambda \perp Z_\lambda \in \mathcal{D}(\lambda)$ and $X_\lambda = Z_\lambda$ and we obtain

$$(4.17) \quad 2(1 + \sqrt{1-f_1+f_3}) - f_4 + f_6 = f_1 + f_5(1 - f_1 + f_3) + 2f_4\sqrt{1-f_1+f_3}.$$

Last, if we take two unit vector fields $X_{-\lambda} \perp Z_{-\lambda} \in \mathcal{D}(-\lambda)$ and $Y_{-\lambda} = Z_{-\lambda}$ in (4.12) then

$$(4.18) \quad 2(1 - \sqrt{1-f_1+f_3}) - f_4 + f_6 = f_1 + f_5(1 - f_1 + f_3) - 2f_4\sqrt{1-f_1+f_3}.$$

Combining and reordering equations (4.13)-(4.18), we get the following compatible system

$$\left. \begin{aligned}
f_1 - f_2 - f_3 - f_4 + f_6 + f_5(1 - f_1 + f_3) &= 0 \\
f_1 + f_2 - f_3 + f_5(1 - f_1 + f_3) &= 0 \\
2f_2 + f_4 - f_6 &= 0 \\
2f_1 + 3f_2 - f_3 + f_4 - f_6 &= 0 \\
f_5(1 - f_1 + f_3) + 2(f_4 - 1)\sqrt{1-f_1+f_3} + f_1 + f_4 - f_6 - 2 &= 0 \\
f_5(1 - f_1 + f_3) + 2(1 - f_4)\sqrt{1-f_1+f_3} + f_1 + f_4 - f_6 - 2 &= 0
\end{aligned} \right\}$$

whose solution is:

$$\begin{aligned}
f_1 &= \frac{f_6 + 1}{2}, & f_2 &= \frac{f_6 - 1}{2}, & f_3 &= \frac{3f_6 + 1}{2}, \\
f_4 &= 1, & f_5 &= \frac{1}{2}, & f_6 &\text{arbitrary.}
\end{aligned}$$

Therefore, $\kappa = f_1 - f_3 = -f_6$, $\mu = f_4 - f_6 = 1 - f_6$ and $c = f_1 + 3f_2 = 2f_6 - 1$, by virtue of Proposition 3.7 and Theorem 4.1.

Now, since κ is constant and less than 1, we have that f_6 is constant and greater than -1 and we achieve the result.

Conversely, given a $(-f_6, 1 - f_6)$ -space of dimension greater than or equal to 5 with ϕ -sectional curvature $c = 2f_6 - 1 > -3$, it follows that f_6 must be a constant function greater than -1 . It is sufficient to apply Theorem 4.3 with $c = 2f_6 - 1$ to get that the manifold has curvature tensor

$$(4.19) \quad R = \frac{f_6 + 1}{2}R_1 + \frac{f_6 - 1}{2}R_2 + \frac{3f_6 + 1}{2}R_3 + R_4 + \frac{1}{2}R_5 + f_6R_6,$$

so that it is a contact metric generalized (κ, μ) -space form $M(f_1, \dots, f_6)$ with functions f_1, \dots, f_6 satisfying equations (4.8). It is obvious that the manifold is non-Sasakian because $\kappa = f_1 - f_3 = -f_6 < 1$. \square

Remark 4.9. We observe that $f_4, f_5 \neq 0$ in (4.8), so there are no examples of non-Sasakian, contact metric generalized Sasakian space forms of dimension greater than or equal to 5 (already seen in [2]).

We will now give a method to construct $(-f_6, 1 - f_6)$ -spaces with constant ϕ -sectional curvature $c = 2f_6 - 1$ for every constant $f_6 > -1$. Due to the previous theorem, they will be examples of contact metric generalized (κ, μ) -space forms.

Let M be a manifold of dimension greater than or equal to 5 and constant sectional curvature $c_s > -1$ ($c_s \neq 1$). Then its tangent sphere bundle with the usual contact metric, $(T_1M, \xi_1, \eta_1, \phi_1, g_1)$, is a (κ, μ) -space with $\kappa = c_s(2 - c_s) \neq 1$ and $\mu = -2c_s < 2$ ([10, Theorem 4]).

By applying a D_a -homothetic deformation we obtain $(T_1M, \bar{\xi}, \bar{\eta}, \bar{\phi}, \bar{g})$, with

$$\bar{\xi} = \frac{1}{a}\xi, \quad \bar{\eta} = a\eta, \quad \bar{\phi} = \phi \quad \text{and} \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where $a > 0$ is a real number.

It is known from [10] that this is a $(\bar{\kappa}, \bar{\mu})$ -space with

$$\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2} \neq 1 \quad \text{and} \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

If we choose $a = (\kappa - 1)/(\mu - 2) > 0$, then $\bar{\mu} = \bar{\kappa} + 1$ and $(T_1M, \bar{\xi}, \bar{\eta}, \bar{\phi}, \bar{g})$ has constant ϕ -sectional curvature $\bar{c} = -(\bar{\kappa} + \bar{\mu})$ because of Theorems 4.3 and 4.4.

Therefore, $\bar{\kappa} = -f_6$, $\bar{\mu} = 1 - f_6$ and $\bar{c} = 2f_6 - 1$ if and only if

$$(4.20) \quad (3 - f_6)c_s^2 + (10 + 2f_6)c_s + (3 - f_6) = 0.$$

If $f_6 = 3$, equation (4.20) has solution $c_s = 0$, which is in particular greater than -1 and not equal to 1. If $f_6 \neq 3$, then (4.20) has the real solutions

$$(4.21) \quad c_s = \frac{-5 - f_6 \pm 4\sqrt{f_6 + 1}}{3 - f_6},$$

which are not equal to 1 because f_6 is greater than -1 . Furthermore, if we consider the positive sign on (4.21), it can be proved that c_s is also greater than -1 . Therefore, we have managed to obtain examples of $(-f_6, 1 - f_6)$ -spaces with the required conditions for every constant function $f_6 > -1$.

Remark 4.10. An alternative proof of Theorem 4.8 follows from [13, Theorem 5], which states that, if M is a non-Sasakian (κ, μ) -space, then its curvature tensor R is given by

$$\begin{aligned}
 R(X, Y, Z, W) = & \left(1 - \frac{\mu}{2}\right) R_1(X, Y, Z, W) - \frac{\mu}{2} R_2(X, Y, Z, W) \\
 & + \left(1 - \frac{\mu}{2} - \kappa\right) R_3(X, Y, Z, W) + R_4(X, Y, Z, W) \\
 (4.22) \quad & + \left(\frac{1 - \frac{\mu}{2}}{1 - \kappa}\right) \{g(hY, Z)g(hX, W) - g(hX, Z)g(hY, W)\} \\
 & + \left(\frac{\kappa - \frac{\mu}{2}}{1 - \kappa}\right) \{g(\phi hY, Z)g(\phi hX, W) - g(\phi hX, Z)g(\phi hY, W)\} \\
 & + (1 - \mu) R_6(X, Y, Z, W).
 \end{aligned}$$

Therefore, if $M(f_1, \dots, f_6)$ is a contact metric generalized (κ, μ) -space form of dimension greater than or equal to 5 satisfying $f_1 - f_3 < 1$, then M is a non-Sasakian (κ, μ) -space with $\kappa = f_1 - f_3 < 1$ and $\mu = f_4 - f_6$ thanks to Theorems 4.1 and 4.5. Comparing (3.1) with (4.22), we obtain a system whose solution is given by (4.8).

What can we say now for 3-dimensional generalized (κ, μ) -space form $M^3(f_1, \dots, f_6)$? First, let us mention that the writing of its curvature tensor is not unique:

Theorem 4.11. *Let M^3 be a contact metric manifold such that its curvature tensor can be simultaneously written as*

$$(4.23) \quad R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6$$

and

$$(4.24) \quad R = f_1^* R_1 + f_2^* R_2 + f_3^* R_3 + f_4^* R_4 + f_5^* R_5 + f_6^* R_6,$$

where $f_1 - f_3 < 1$. Then the functions f_i and f_i^* , $i = 1, \dots, 6$, are related as follows,

$$\begin{aligned}
 (4.25) \quad f_1^* &= f_1 + f, & f_2^* &= f_2 - f/3, & f_3^* &= f_3 + f, \\
 f_4^* &= f_4 + \bar{f}, & f_6^* &= f_6 + \bar{f},
 \end{aligned}$$

where f and \bar{f} are arbitrary functions on M .

Proof. We know that the manifold is in particular a generalized (κ, μ) -space with $\kappa = f_1 - f_3 < 1$ and $\mu = f_4 - f_6$ (Theorem 4.1). Therefore, we can consider a ϕ -basis $\{X, \phi X, \xi\}$ with $X \in D(\lambda)$, where $\lambda = \sqrt{1 - \kappa} = \sqrt{1 - (f_1 - f_3)} > 0$ (Theorem 4.2). By virtue of Propositions 3.7, 3.8 and Corollary 3.9, if we calculate $K(X, \xi)$, $K(\phi X, \xi)$ and $K(X, \phi X)$ by using both (4.23) and (4.24) we obtain the system

$$\begin{cases} (f_1^* - f_1) - (f_3^* - f_3) &= 0 \\ (f_4^* - f_4) - (f_6^* - f_6) &= 0 \\ (f_1^* - f_1) + 3(f_2^* - f_2) &= 0 \end{cases}$$

whose general solution is given by (4.25). \square

Remark 4.12. In the conditions of the previous theorem, if $\kappa = f_1 - f_3 = 1$, then M^3 is a Sasakian manifold and therefore it is a generalized Sasakian space form $M(f_1, f_2, f_3)$. In [2], P. Alegre and A. Carriazo proved that in such a case the functions f_i and f_i^* , $i = 1, 2, 3$, are related as in (4.25).

The converse of Theorem 4.11 is also true:

Theorem 4.13. *Let $M^3(f_1, \dots, f_6)$ be a contact metric generalized (κ, μ) -space form. If we define the functions $f_1^*, f_2^*, f_3^*, f_4^*, f_6^*$ as in (4.25), for certain functions f, \bar{f} on M , and we take an arbitrary function f_5^* , then M^3 is also a generalized (κ, μ) -space form $M^3(f_1^*, \dots, f_6^*)$.*

Proof. It follows from (3.1) and (4.25) that the curvature tensor satisfies

$$R = \sum_{i=1}^6 f_i R_i = \sum_{i=1}^6 f_i^* R_i + f \left(-R_1 + \frac{1}{3}R_2 - R_3 \right) - \bar{f}(R_4 + R_6) + (f_5 - f_5^*)R_5.$$

To obtain (4.24) it is enough to check that the last terms vanish, which is true because

$$-R_1 + \frac{1}{3}R_2 - R_3 = R_4 + R_6 = R_5 = 0$$

for every 3-dimensional contact metric manifold. \square

Therefore, if $M^3(f_1, \dots, f_6)$ is a contact metric generalized (κ, μ) -space form, its curvature tensor can be written as

$$(4.26) \quad R = f_1^* R_1 + f_3^* R_3 + f_4^* R_4,$$

so M^3 is also $M(f_1^*, 0, f_3^*, f_4^*, 0, 0)$ for $f_1^* = f_1 + 3f_2, f_3^* = f_3 + 3f_2, f_4^* = f_4 - f_6$. In order to consider a unique writing of the curvature tensor of a 3-dimensional, contact metric generalized (κ, μ) -space form, we will choose R satisfying $f_2^* = f_5^* = f_6^* = 0$.

D. E. Blair, T. Koufogiorgos and B. J. Papantoniou classified the 3-dimensional (κ, μ) -spaces in [10, Theorem 3]. Using that result and Theorems 4.1 and 4.2 we get:

Theorem 4.14. *Let $M^3(f_1, \dots, f_6)$ be a contact metric generalized (κ, μ) -space form. If $f_1 - f_3 \neq 1$ is constant, then $f_4 - f_6$ is also constant, M satisfies*

$$(4.27) \quad 2f_1 + 3f_2 - f_3 + f_4 - f_6 = 0,$$

and it is locally isometric to one of the following Lie groups with a left invariant metric: $SU(2)$ (or $SO(3)$), $SL(2, \mathbf{R})$ (or $O(1, 2)$), $E(2)$ (the group of rigid motions of the Euclidean 2-space) or $E(1, 1)$ (the group of rigid motions of the Minkowski 2-space).

Moreover this structure can occur on:

- $SU(2)$ or $SO(3)$ if $1 - \lambda - \frac{\mu}{2} > 0$ and $1 + \lambda - \frac{\mu}{2} > 0$,
- $SL(2, \mathbf{R})$ or $O(1, 2)$ if $\begin{cases} 1 - \lambda - \frac{\mu}{2} < 0 & \text{and} & 1 + \lambda - \frac{\mu}{2} > 0 \\ 1 - \lambda - \frac{\mu}{2} < 0 & \text{and} & 1 + \lambda - \frac{\mu}{2} < 0, \end{cases}$
- $E(2)$ if $1 - \lambda - \frac{\mu}{2} = 0$ and $\mu < 2$,
- $E(1, 1)$ if $1 + \lambda - \frac{\mu}{2} = 0$ and $\mu > 2$,

where $\kappa = f_1 - f_3, \lambda = \sqrt{1 - \kappa}$ and $\mu = -2f_1 - 3f_2 + f_3$.

Proof. Theorems 3.4 and 4.1 imply that M is a non-Sasakian generalized (κ, μ) -space with $\kappa = f_1 - f_3$ and $\mu = f_4 - f_6$. Since $f_1 - f_3$ is constant, then we know from Theorem 4.7 that $f_4 - f_6$ is also constant and hence M is a (κ, μ) -space. Applying Theorem 4.2 we obtain that R must satisfy the six equations (4.1)-(4.6).

If we take unit vector fields $Y_{-\lambda} = Z_{-\lambda}$ and $X_\lambda = \phi Y_{-\lambda}$ in (4.3), then

$$-(f_1 - f_3) - (f_4 - f_6) = f_1 + 3f_2.$$

A short calculation yields $2f_1 + 3f_2 - f_3 + f_4 - f_6 = 0$ and $\mu = -2f_1 - 3f_2 + f_3$. We only have to apply [10, Theorem 3] to end the proof. \square

Remark 4.15. We notice that the different cases of Theorem 4.14 depend on the value of κ and μ , which are determined only by the functions f_1, f_2 and f_3 and do not depend explicitly on the functions f_4, f_5 or f_6 .

Let us recall that a contact metric manifold is said to be η -Einstein [9, p. 105] if it satisfies

$$Q = aI + b\eta \otimes \xi,$$

where a, b are some differentiable functions on M .

In [11], D. E. Blair, T. Koufogiorgos and R. Sharma studied the contact metric manifolds satisfying $Q\phi = \phi Q$ and obtained the following result:

Proposition 4.16. *If $(M^3, \phi, \xi, \eta, g)$ is a contact metric manifold, then the following statements are equivalent:*

- (i) M is η -Einstein,
- (ii) $Q\phi = \phi Q$, where Q is the Ricci operator,
- (iii) M is a $(\kappa, 0)$ -space, with κ constant.

In view of Theorem 4.1, we know that a generalized (κ, μ) -space form $M(f_1, \dots, f_6)$ with contact metric structure satisfies condition (iii) if and only if $f_1 - f_3$ is constant and $f_4 - f_6 = 0$. We will now study when the conditions (i) and (ii) hold. First, we calculate the Ricci operator Q :

Proposition 4.17. *If $M^{2n+1}(f_1, \dots, f_6)$ is a contact metric generalized (κ, μ) -space form, then*

$$(4.28) \quad Q = (2nf_1 + 3f_2 - f_3)I - (3f_2 + (2n-1)f_3)\eta \otimes \xi + ((2n-1)f_4 - f_6)h.$$

Moreover, if we also suppose that $\kappa = f_1 - f_3 \neq 1$ is constant, then M is a non-Sasakian (κ, μ) -space and

$$(4.29) \quad Q = (2(n-1) - n\mu)I + (2(1-n) + n(2\kappa + \mu))\eta \otimes \xi + (2(n-1) + \mu)h.$$

Proof. A straightforward computation with respect to a ϕ -basis gives (4.28). On the other hand, if $\kappa = f_1 - f_3 \neq 1$ is constant, then $\mu = f_4 - f_6$ is also constant by Theorem 4.7. Therefore, the equations (4.8) hold if M has dimension greater than or equal to 5 and (4.27) is true if M is a 3-dimensional manifold. Applying them to (4.28) yields (4.29). \square

Remark 4.18. Let us notice that the previous proposition means that a contact metric generalized (κ, μ) -space form $M^{2n+1}(f_1, \dots, f_6)$ is η -Einstein if and only if $f_4(2n-1) - f_6 = 0$. In particular, $M^3(f_1, \dots, f_6)$ satisfies condition (i) if and only if $f_4 - f_6 = 0$.

We can also prove the following:

Proposition 4.19. *If $M^{2n+1}(f_1, \dots, f_6)$ is a contact metric generalized (κ, μ) -space form, then*

$$(4.30) \quad Q\phi - \phi Q = 2((2n-1)f_4 - f_6)h\phi,$$

where Q denotes the Ricci operator on M .

Proof. Using $\eta \circ \phi = 0$, from (4.28) we obtain

$$(4.31) \quad Q\phi = (2nf_1 + 3f_2 - f_3)\phi + ((2n-1)f_4 - f_6)h\phi.$$

Applying ϕ to (4.28) and using (2.3) we get

$$(4.32) \quad \phi Q = (2nf_1 + 3f_2 - f_3)\phi - ((2n-1)f_4 - f_6)h\phi.$$

Therefore, (4.31) and (4.32) imply (4.30). \square

Remark 4.20. We deduce from Proposition 4.19 that if $M^{2n+1}(f_1, \dots, f_6)$ is a contact metric generalized (κ, μ) -space form, then $Q\phi = \phi Q$ is true if and only if $(2n-1)f_4 - f_6 = 0$. In particular, M^3 satisfies condition $Q\phi = \phi Q$ if and only if $f_4 - f_6 = 0$.

We may resume the previous results in this proposition:

Proposition 4.21. *If $M^3(f_1, \dots, f_6)$ is a contact metric generalized (κ, μ) -space form, then the following conditions are equivalent:*

- (i) M^3 is η -Einstein,
- (ii) $Q\phi = \phi Q$, where Q denotes the Ricci operator,
- (iii) M^3 is a $(f_1 - f_3, 0)$ -space,
- (iv) $f_4 - f_6 = 0$.

Remark 4.22. If a contact metric generalized (κ, μ) -space $M^3(f_1, \dots, f_6)$ satisfies $f_4 - f_6 = 0$, then $f_1 - f_3$ is constant. This would be a particular case of [26, Theorem 10].

D. E. Blair and H. Chen proved in [12] the following theorem, which improves the classification of contact metric manifolds satisfying $Q\phi = \phi Q$ given in [11]:

Theorem 4.23. *Let M^3 be a contact metric manifold satisfying $Q\phi = \phi Q$. Then M^3 is either Sasakian, flat or locally isometric to a left invariant metric on the Lie group $SU(2)$ or $SL(2, \mathbf{R})$. In the latter case M^3 has constant ξ -sectional curvature $\kappa < 1$ and constant ϕ -sectional curvature $-\kappa$ (this structure can occur on $SU(2)$ if $\kappa > 0$ and on $SL(2, \mathbf{R})$ if $\kappa < 0$).*

We deduce from the previous theorem the following proposition:

Proposition 4.24. *Let $M^3(f_1, \dots, f_6)$ be a contact metric generalized (κ, μ) -space form. If $f_1 - f_3 \neq 1$ and $f_4 - f_6 = 0$, then $f_1 - f_3$ and $f_1 + 3f_2$ are constant and $2f_1 + 3f_2 - f_3 = 0$ holds.*

Proof. We know that M^3 is a contact metric manifold satisfying $Q\phi = \phi Q$ due to Proposition 4.21. Applying Theorem 4.23 and using the ϕ -sectional and ξ -sectional formulas from Propositions 3.7 and 3.8 we get the wanted result. \square

We will now study the scalar curvature:

Proposition 4.25. *If $M^3(f_1, \dots, f_6)$ is a contact metric generalized (κ, μ) -space form, then the scalar curvature τ is given by*

$$(4.33) \quad \tau = 2(3f_1 + 3f_2 - 2f_3).$$

Moreover, if $\kappa = f_1 - f_3 \neq 1$ is constant, then

$$\tau = 2(\kappa - \mu).$$

Proof. A straightforward computation with respect to a ϕ -basis yields

$$\tau = \text{tr}(Q) = 2(K(X, \phi X) + K(X, \xi) + K(\phi X, \xi)),$$

and using Propositions 3.7, 3.8 and Corollary 3.9 we get (4.33).

Furthermore, if $\kappa = f_1 - f_3 \neq 1$ is constant, then Theorem 4.7 implies that $\mu = f_4 - f_6$ is also constant and we only need to apply (4.27) to obtain $3f_1 + 3f_2 - 2f_3 = (f_1 - f_3) - (f_4 - f_6) = \kappa - \mu$. \square

Corollary 4.26. *Let $M^3(f_1, \dots, f_6)$ be a contact metric generalized (κ, μ) -space form with $\kappa = f_1 - f_3 \neq 1$ constant. Then*

$$Q = \left(\frac{\tau}{2} - \kappa\right) I + \left(3\kappa - \frac{\tau}{2}\right) \eta \otimes \xi + \left(\kappa - \frac{\tau}{2}\right) h.$$

We will prove the next theorem using the formulas we have obtained for Q and τ and the expression that connects both of them to R on a 3-dimensional Riemannian manifold:

$$(4.34) \quad \begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y \\ &\quad - \frac{\tau}{2}\{g(Y, Z)X - g(X, Z)Y\} \end{aligned}$$

for any vector fields X, Y, Z on M .

Theorem 4.27. *If $M^3(f_1, \dots, f_6)$ is a contact metric generalized (κ, μ) -space form, then its curvature tensor can be written as:*

$$(4.35) \quad R = (f_1 + 3f_2)R_1 + (3f_2 + f_3)R_3 + (f_4 - f_6)R_4.$$

If we also suppose that $\kappa = f_1 - f_3 \neq 1$ is constant, then

$$R = -(\kappa + \mu)R_1 - (2\kappa + \mu)R_3 + \mu R_4.$$

Proof. Equation (4.35) is obtained by substituting (4.28) and (4.33) in (4.34). Moreover, if $\kappa = f_1 - f_3 \neq 1$ is constant, then we obtain from Theorem 4.7 that $\mu = f_4 - f_6$ is also constant. Therefore, (4.27) holds, which yields $f_1 + 3f_2 = -(\kappa + \mu)$ and $3f_2 + f_3 = -(2\kappa + \mu)$. \square

We will now check that the first example of a generalized (κ, μ) -space of dimension 3 given by T. Koufogiorgos and C. Tsihlias in [22] is a contact metric generalized (κ, μ) -space form $M^3(f_1^*, 0, f_3^*, f_4^*, 0, 0)$ with f_1^*, f_3^*, f_4^* not constant.

Let M^3 be the manifold $M = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_3 \neq 0\}$, where (x_1, x_2, x_3) are the standard coordinates on \mathbf{R}^3 . The vector fields

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = -2x_2x_3 \frac{\partial}{\partial x_1} + \frac{2x_1}{x_3^3} \frac{\partial}{\partial x_2} - \frac{1}{x_3^2} \frac{\partial}{\partial x_3}, \quad e_3 = \frac{1}{x_3} \frac{\partial}{\partial x_2}$$

are linearly independent at each point of M . We get

$$[e_1, e_2] = \frac{2}{x_3^2} e_3, \quad [e_2, e_3] = 2e_1 + \frac{1}{x_3^3} e_3, \quad [e_3, e_1] = 0.$$

Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, 2, 3$, ∇ its Riemannian connection, R the curvature tensor of g and η the 1-form defined by $\eta(X) = g(X, e_1)$, for any X on M , which is a contact form because $\eta \wedge d\eta \neq 0$. Let ϕ be the $(1, 1)$ -tensor field defined by

$$\phi e_1 = 0, \quad \phi e_2 = e_3, \quad \phi e_3 = -e_2.$$

Using the linearity of $\phi, d\eta$ and g we find

$$\begin{aligned}\eta(e_1) &= 1, \quad \phi^2 X = -X + \eta(X) e_1, \quad d\eta(X, Y) = g(X, \phi Y), \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X) \eta(Y),\end{aligned}$$

for any vector fields X, Y on M . Hence (ϕ, e_1, η, g) defines a contact metric structure on M . Using Koszul's formula we obtain:

$$\begin{aligned}\nabla_{e_1} e_2 &= \left(-1 + \frac{1}{x_3^2}\right) e_3, \quad \nabla_{e_2} e_1 = -\left(1 + \frac{1}{x_3^2}\right) e_3 \\ \nabla_{e_1} e_3 &= \left(1 - \frac{1}{x_3^2}\right) e_2, \quad \nabla_{e_3} e_1 = \left(1 - \frac{1}{x_3^2}\right) e_2, \\ \nabla_{e_2} e_3 &= \left(1 + \frac{1}{x_3^2}\right) e_1, \quad \nabla_{e_3} e_2 = \left(-1 + \frac{1}{x_3^2}\right) e_1 - \frac{1}{x_3^3} e_3, \\ \nabla_{e_2} e_2 &= 0, \quad \nabla_{e_3} e_3 = \frac{1}{x_3^3} e_2.\end{aligned}$$

The tensor h satisfies

$$he_1 = 0, \quad he_2 = \lambda e_2, \quad he_3 = -\lambda e_3,$$

where $\lambda = \frac{1}{x_3^2}$. Putting $\mu = 2\left(1 - \frac{1}{x_3^2}\right)$ and $\kappa = 1 - \frac{1}{x_3^4}$, we obtain

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

Therefore M is a generalized (κ, μ) -space with κ, μ non-constant functions on M .

Let us now see that the manifold M is also a generalized (κ, μ) -space form $M(f_1^*, 0, f_3^*, f_4^*, 0, 0)$. From the definition of Riemannian curvature, we get that the non-trivial curvatures are:

$$\begin{aligned}R(e_1, e_2)e_1 &= -(\kappa + \lambda\mu)e_2, \\ R(e_1, e_2)e_2 &= (\kappa + \lambda\mu)e_1, \\ R(e_1, e_2)e_3 &= 0, \\ R(e_1, e_3)e_1 &= (-\kappa + \lambda\mu)e_3, \\ R(e_1, e_3)e_2 &= 0, \\ R(e_1, e_3)e_3 &= (\kappa - \lambda\mu)e_1, \\ R(e_2, e_3)e_1 &= 0, \\ R(e_2, e_3)e_2 &= (\kappa + \mu - 2\lambda^3)e_3, \\ R(e_2, e_3)e_3 &= -(\kappa + \mu - 2\lambda^3)e_2.\end{aligned}$$

On the other hand, we know that every contact metric generalized (κ, μ) -space form $M(f_1^*, 0, f_3^*, f_4^*, 0, 0)$ is a generalized (κ^*, μ^*) -space with $\kappa^* = f_1^* - f_3^*$ and $\mu^* = f_4^*$. Using (4.26), we can write the above curvature values in terms of κ^*, μ^* and f_1^* . If we make equal both sets of equations, we obtain a system which can be simplified to:

$$\begin{cases} \kappa &= \kappa^* = f_1^* - f_3^* \\ \mu &= \mu^* = f_4^* \\ f_1^* &= -(\kappa + \mu - 2\lambda^3). \end{cases}$$

Therefore, the solution to the system is:

$$\begin{cases} f_1^* &= -3 + \frac{2}{x_3^2} + \frac{1}{x_3^4} + \frac{2}{x_3^6} \\ f_3^* &= -4 + \frac{2}{x_3^2} + \frac{2}{x_3^4} + \frac{2}{x_3^6} \\ f_4^* &= 2 \left(1 - \frac{1}{x_3^2} \right). \end{cases}$$

Thus we conclude that this example is a contact metric generalized (κ, μ) -space form $M^3(f_1^*, 0, f_3^*, f_4^*, 0, 0)$, where f_1^*, f_3^*, f_4^* are non-constant functions. Moreover, its scalar curvature is, by (4.33),

$$\tau = 2(3f_1^* - 2f_3^*) = 2 \left(-1 + \frac{2}{x_3^2} - \frac{1}{x_3^4} + \frac{2}{x_3^6} \right),$$

and therefore not constant.

Remark 4.28. The second example of generalized (κ, μ) -space of dimension 3 given by T. Koufogiorgos and C. Tsihlias in [22] is also a contact metric generalized (κ, μ) -space form $M^3(f_1^*, 0, f_3^*, f_4^*, 0, 0)$ with non-constant functions:

$$\begin{cases} f_1^* &= -3 - \frac{2}{x_3^4} + \frac{1}{x_3^8} + \frac{10}{x_3^{14}} \\ f_3^* &= -4 - \frac{2}{x_3^4} + \frac{2}{x_3^8} + \frac{10}{x_3^{14}} \\ f_4^* &= 2 \left(1 + \frac{1}{x_3^4} \right). \end{cases}$$

5. TRANS-SASAKIAN GENERALIZED (κ, μ) -SPACE FORMS

We will see in this section that if a manifold is trans-Sasakian, then $h = 0$. Hence every generalized (κ, μ) -space form $M(f_1, \dots, f_6)$ with a trans-Sasakian structure is a generalized Sasakian space form (see [1, 2]).

We recall that an almost contact metric manifold (M, ϕ, ξ, η, g) is said to be *trans-Sasakian* if there exist functions α and β on M such that

$$(5.1) \quad (\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),$$

for any vector fields X, Y on M . In a trans-Sasakian manifold it is known that

$$(5.2) \quad \nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi).$$

We will now prove two properties that trans-Sasakian manifolds have in common with contact metric manifolds.

Proposition 5.1. *If (M, ϕ, ξ, η, g) is a trans-Sasakian manifold, then $\nabla_\xi \phi = 0$ and $\nabla_\xi \xi = 0$.*

Proof. It follows directly from (5.1) and (5.2). □

Proposition 5.2. *If (M, ϕ, ξ, η, g) is a trans-Sasakian manifold, then $h = 0$.*

Proof. Using the definition of h and applying the usual properties we obtain

$$2hX = (L_\xi \phi)X = (\nabla_\xi \phi)X - \nabla_{\phi X} \xi + \phi \nabla_X \xi.$$

Therefore, the result follows directly from (5.2) and Proposition 5.1. □

Corollary 5.3. *Every trans-Sasakian manifold with a contact metric structure is Sasakian.*

Proof. If M is a trans-Sasakian manifold, it follows that $h = 0$ from Proposition 5.2. If M is also a contact metric manifold, then it is a K -contact manifold and $\nabla_X \xi = -\phi X$ is satisfied. Comparing such an equation with (5.2), we deduce that $\alpha = 1$ and $\beta = 0$. Substituting these values in (5.1), we obtain (2.2), which is one of the characterizations of Sasakian manifolds. \square

In view of Proposition 5.2, we have the following

Theorem 5.4. *Every trans-Sasakian generalized (κ, μ) -space form is a generalized Sasakian space form.*

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